Breakdown of superfluidity of a matter wave in a random environment

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We consider a guided Bose-Einstein matter wave flowing through a disordered potential. We determine the critical velocity at which superfluidity is broken and compute its statistical properties. They are shown to be connected to extreme values of the random potential. Experimental implementations of this physics are discussed.

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I. INTRODUCTION

The simplest and most intuitive definition of superfluidity (SF) is the ability to move without dissipation. According to a perturbative mechanism proposed by Landau, superfluidity is broken when the velocity of the flow exceeds a critical value $V_c^{\rm L}$ at which it is energetically favorable to emit elementary excitations. Though this mechanism has been explicitly verified in ^{4}He [1], ^{3}He-B [2] and Bose Einstein condensates (BECs) [3], many experiments in ^{4}He [4], ^{3}He-A [5] and BECs [6, 7] have shown that the actual critical velocity V_c is generally lower than $V_c^{\rm L}$ due to the occurrence of phase slips. In this scenario, SF is protected by an energy barrier which may be overcome by fluctuations of thermal (as first suggested by Iordanskii in the context of liquid He-II [8] and by Little for superconductors [9]) or quantal origin (as more recently observed in ^{4}He [10], in superconducting nano-wires [11] and possibly also in BECs [12]), leading to what is called a resistive state in the physics of superconductors.

In what follows we address the problem of determining the critical velocity for the breakdown of SF of a matter wave moving in a disordered potential. The matter wave beam is formed by a guided BEC in a quasi one-dimensional (1D) geometry. We assume zero temperature; it is well known that in this case superconductivity and superfluidity are not destroyed by weak disorder. This is Anderson's theorem for non-magnetic impurities in superconductors [13]; similar results (with a different physical mechanism) hold for BECs [14]. In the latter case, the phase coherence of the system is preserved in 1D in the presence of a weak disorder as demonstrated in the experiments reported in Refs. [15, 16]. As a consequence, one may study a simple scenario for breakdown of superfluidity in disordered BECs where phase slips are neither thermally nor quantum mechanically nucleated but rather have a dynamical origin : the barrier disappears at a given critical velocity. This mechanism is standard in the absence of disorder (see, e.g., [17] and references therein) and the extraordinary control achieved in the domain of atomic vapor has even allowed a direct observation of the nonlinear excitations nucleated above the critical velocity V_c [6, 7]. However, to our knowledge there is up to now only one clear experimental evidence of dynamical breakdown of SF and of finite critical velocity in the presence of disorder, obtained by studying the damping of dipole oscillations in an elongated BEC [18]. In our fully 1D case, as well as in the dipole oscillation experiments [18], an important issue is to understand the out-of equilibrium solutions of a nonlinear continuous system in the presence of disorder. In this context, the phase diagram of the fluid flowing through a quasi 1D disordered potential $U(x)$ of finite extent L was recently studied in Refs. [19–21]. Here, we concentrate on the SF part of this diagram and more specifically on the description of the breakdown of SF when the velocity (or the length L of the disordered region) increases.

We study two different types of disordered potential with opposite characteristics. The first one is a smooth potential whose typical spatial scale of variation is large compared to the healing length of the condensate. In this case a local density approximation holds and a local Landau criterion can be applied [22]. This mechanism reconciles the Landau approach with the phase slip phenomenon, because it predicts that SF is broken when the local Landau velocity is reached by emission of nonlinear excitations (solitons in our 1D case). The second type of disordered potential consists in a series of point-like impurities and thus has, contrarily to the previous type, strong fluctuations on small spatial scales. We show that in this case the criterion can be adapted yielding – as in the previous case – very good agreement with numerical simulations. In both cases we explicitly compute the statistical properties of the critical velocity V_c and show that they are closely related to the extreme value statistics of the disordered potential. We finally discuss experimental realizations of our models.

The system considered is a weakly interacting BEC transversely confined by a harmonic potential of frequency $\omega_\perp$. For simplicity, the disordered potential U is supposed to depend on a single spatial variable – the coordinate x along the axial direction of the guide. A stationary flow of the system is then accurately described by a 1D order parameter $\psi(x)$ obeying the nonlinear Schrödinger equation [23, 24]

$$\mu\,\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left[U(x) + g\,n^\nu(x)\right]\psi\ . \qquad (1)$$

Here, $n(x) \equiv |\psi(x)|^2$ is the condensate density per unit

of longitudinal length, μ is the chemical potential and $g = 2\hbar\omega_\perp a^\nu$ is the nonlinear parameter ($a > 0$ is the 3D s-wave scattering length). In the low density regime ($an \ll 1$) the density profile in the transverse direction is Gaussian-shaped and $\nu = 1$, whereas $\nu = 1/2$ in the opposite high density regime ($an \gg 1$) where the Thomas-Fermi approximation holds for the transverse degree of freedom [25].

II. SUPERFLUID FLOWS

In addition to the density $n(x)$, it is convenient to characterize the flow by its velocity $v(x) = \frac{\hbar}{m}\,[\arg(\psi)]_x$. We assume that the disordered potential $U(x)$ takes sizable values only over a region of finite length L. In this case a SF flow corresponds to a solution of Eq. (1) with constant density n_0 and velocity V at $\pm\infty$. The chemical potential μ then reads:

$$\mu = \frac{1}{2} m V^2 + g\, n_0^\nu \; . \tag{2}$$

In the following we denote the chemical potential of a BEC at rest as μ_0 ($\mu_0 = g\, n_0^\nu$).

In absence of external potential, the SF solution corresponds to $n(x) = n_0$ and $v(x) = V$ for all x, and one can show that it is stable under a weak perturbing potential provided $V \le c_0$ where

$$c_0 = \sqrt{\frac{\nu\, g\, n_0^\nu}{m}} \tag{3}$$

is the sound velocity of the unperturbed condensate. The condition $V \le c_0$ is exactly the Landau criterion for SF because in a BEC the velocity $V_c^{\rm L}$ is precisely the speed of sound.

III. SLOWLY VARYING DISORDERED POTENTIALS

In the case of a slowly varying potential, that is, when the typical length of spatial variations of $U(x)$ is much larger than the healing length $\xi = \hbar/\sqrt{m\mu_0}$ of the fluid, one can devise a local density approximation for describing stationary flows. In this scheme, the flow verifies current conservation and local equilibrium. This reads $n(x)v(x) = C^{\rm st} = n_0 V$ and

$$\mu = \frac{1}{2} m v^2(x) + g\, n^\nu(x) + U(x) \; . \tag{4}$$

From these considerations it is easy to see that the velocity $v(x)$ is determined by a simple algebraic equation [22]: $U(x) = \mu_0\, G(v(x))$ where $G(v) = 1 - (V/v)^\nu + \frac{\nu}{2}(V^2 - v^2)/c_0^2$. This equation admits a solution provided $U(x)/\mu_0$ is lower than the maximum of G. If this condition is violated the flow no longer admits a stationary solution, SF is broken and the flow becomes dissipative (one can show that in this case the obstacle described by the potential $U(x)$ experiences a finite drag [26]). The maximum of G is reached when $V^\nu c_0^2 = [v(x)]^{2+\nu}$, which precisely reads $v(x) = c(x)$ where $c(x) = \sqrt{\nu g\, n^\nu(x)/m}$ can be termed the local sound velocity [compare with (3)] : SF is broken when one reaches the local Landau criterion.

In this approximation, it is clear that SF will first break down at the point $x = x_{\rm m}$ where the potential is maximum: $U(x_{\rm m}) = \max\,[U(x)] = U_{\rm m}$. The condition $v(x_{\rm m}) = c(x_{\rm m})$ yields an explicit relation between $U_{\rm m}$ and V_c:

$$\frac{U_{\rm m}}{\mu_0} = 1 + \frac{\nu}{2}\left(\frac{V_c}{c_0}\right)^2 - \left(1 + \frac{\nu}{2}\right)\left(\frac{V_c}{c_0}\right)^{2\nu/(\nu+2)} \; . \tag{5}$$

Everything now boils down to a problem of extreme value: once the statistical properties of the maximum $U_{\rm m}$ of $U(x)$ over $[0, L]$ are known, the distribution of the critical velocities V_c can be obtained readily through (5). Namely, if one denotes by $\mathscr{P}_L(U_{\rm m})$ the probability distribution of $U_{\rm m}$ and by $P_L(V_c)$ the corresponding distribution of the critical velocity V_c one has

$$P_L(V_c) = \frac{\nu\mu_0}{c_0}\left[\frac{V_c}{c_0} - \left(\frac{V_c}{c_0}\right)^{(\nu-2)/(\nu+2)}\right] \mathscr{P}_L(U_{\rm m}) \; . \tag{6}$$

The distribution function of $U_{\rm m}$ (and thus also that of V_c) depends on the size L of the disordered region for the simple reason that the longer the disordered region, the larger the probability of finding a large maximum of U. Hence it is clear that, on average, the critical velocity decreases with increasing sample length L. Note also that, in this picture, the critical velocity is related to the local fluctuations of the disorder and not to the details of the correlations.

In order to calculate the distribution of $U_{\rm m}$, the first step consists in mapping the problem of finding the maximum value of a correlated continuous function to a problem of a set of N discrete uncorrelated variables. According to [27] this mapping can be done if the correlation function of $U(x)$, characterized by a correlation length ℓ_c, decays faster than a logarithm and provided $L/\ell_c \gg 1$. From now on we assume that these requirements are fulfilled, and study the extreme value statistics of a set of $N = \gamma L/\ell_c$ uncorrelated random variables $\{U_1, U_2, ..., U_N\}$ distributed according to the probability distribution of the disorder potential $p(U)$. γ is a parameter of order unity that depends on the correlation function and has to be determined numerically [28]. If we denote by $f(U)$ and $\mathscr{F}_L(U_{\rm m})$ the cumulative distribution functions of U and $U_{\rm m}$, respectively, we have $\mathscr{F}_L(U_{\rm m}) = [f(U_{\rm m})]^N$. Taking the derivative of this expression yields the probability distribution $\mathscr{P}_L(U_{\rm m})$ and then the distribution of the critical velocity through (5) and (6).

To estimate the frontier between the superfluid and the dissipative regimes, we calculate the median $L(V_c)$ of

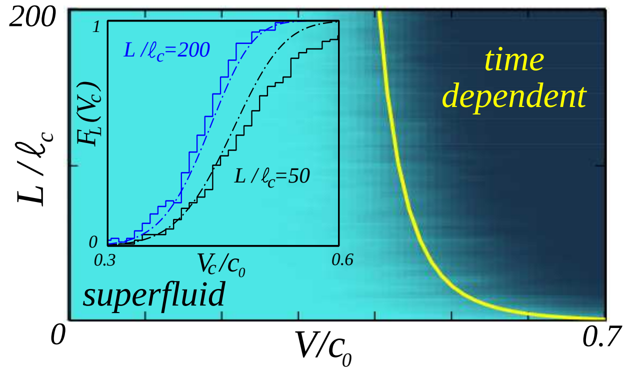

FIG. 1: (Color online) Transport of a quasi 1D BEC with velocity V through a Gaussian correlated disordered potential of extension L (the parameters ν, ℓ_c and Σ take here the values $\nu = 1$, $\ell_c = 5\xi$ and $\Sigma = 0.1\mu_0$). Dark region: time dependent flow; light gray (light blue online) region: SF stationary flow. The (yellow online) solid line displays the boundary between the two regions as predicted by Eq. (7). The inset displays the cumulative probability distribution of the critical velocity V_c for samples of two different lengths. Staircase functions: numerical computations; dot-dashed lines: theory.

this distribution and obtain

$$\frac{L(V_c)}{\ell_c} = \frac{\gamma^{-1}\ln 1/2}{\ln f(U_{\rm m}(V_c))}\,, \tag{7}$$

where $U_{\rm m}(V_c)$ is given by (5).

In order to check the validity of our approach, we have numerically determined the critical velocity from time dependent simulations of the Gross-Pitaevskii equation. Starting from the ground state in the presence of disorder at zero velocity, we have adiabatically accelerated the disordered potential until it reaches a velocity V. For each V and L we consider 80 realizations of the random potential and determine the fraction P_s of stationary solutions. This quantity is plotted in Fig. 1 using a gray scale [dark, $P_s = 0$; light blue (gray), $P_s = 1$] as a function of the normalized variables L/ℓ_c and V/c_0. The inset displays $F_L(V_c)$, the cumulative probability distribution of the critical velocity. Very good agreement is observed in the expected limit of validity of our approach ($L/\ell_c \gg 1$). Figure 1 is drawn in the case of a Gaussian disorder, $p(U) = \exp(-U^2/2\Sigma^2)/\sqrt{2\pi\Sigma^2}$, with a Gaussian correlation function (numerically we find $\gamma \simeq 0.8$). We have also checked the accuracy of our predictions for other types of disorder of experimental interest (Lorentz-correlated disorder and speckle potential).

IV. A SERIES OF δ SCATTERERS

Another commonly used model of disorder is a potential formed by a series of δ-like impurities: $U(x) = \lambda\mu_0\xi\sum_{i=1}^{N}\delta(x-x_i)$, where the x_i's are uncorrelated random variables distributed between 0 and L with density $\rho = N/L$. $\lambda > 0$ is the dimensionless strength of a scatterer. In the presence of such a potential, the local Landau criterion is no longer applicable because the density is not smooth. However one can devise an approach adapted to this particular case. One first remarks that each impurity repels the condensate, the density of which reaches its lowest local value at the position of the δ peak. The region in space where the decrease in density is the largest will correspond to configurations where two (or more) scatterers lie very close to each other. SF will break down, therefore, at the point where the local concentration of δ peaks is maximum.

The smallest length scale for density modulations of the condensate is the healing length ξ. The condensate is therefore not sensitive to details of the disordered potential on scales smaller than ξ. One thus divides the disordered region into $B = L/\xi$ boxes, each of size ξ, and replaces the m_i delta peaks present in box i by a single effective peak of strength $m_i\,\lambda$. The goal is then to determine the probability distribution of intensity of the strongest of the effective peaks, because it is at this peak that SF will first be broken. One thus needs to calculate the probability distribution of $M = \max\{m_1, m_2, ..., m_B\}$, where $\sum_i m_i = N$.

The probability of finding m peaks in an interval of length ξ is $\pi(m) = \binom{N}{m}p^m(1-p)^{N-m}$, where $p = \xi/L$. In the limit of a wide disordered region ($L \to \infty$), the product pN remaining constant, the binomial law can be approximated by a Poisson law of parameter $\zeta = pN$: $\pi(m) \simeq e^{-\zeta}\zeta^m/m!$. In this limit the m_i's are uncorrelated and the cumulative distribution of M is $\mathscr{F}_L(M) = f(M)^B$ where $f(M) = \Gamma(M+1,\zeta)/M!$ is the cumulative distribution function associated with the Poisson law [$\Gamma(x,\zeta)$ is the incomplete gamma function].

One can relate the SF critical velocity to the strength $\Lambda_{\rm e} = M\,\lambda$ of the strongest effective peak by using the criterion for the critical velocity obtained in [23] for a single peak: $\Lambda_{\rm e} = K(V_c/c_0)$, where

$$K(z) = \frac{\sqrt{2}}{4z}\left\{-8z^4 - 20z^2 + 1 + (1+8z^2)^{3/2}\right\}^{1/2}\,. \tag{8}$$

The probability distribution of V_c is then

$$P_L(V_c) = \frac{K'(V_c/c_0)}{\lambda\,c_0}\,\mathscr{P}_L(M = \Lambda_{\rm e}(V_c)/\lambda)\,, \tag{9}$$

where $\mathscr{P}_L(M) = d\mathscr{F}_L(M)/dM$. Defining again the typical critical velocity as the median, the boundary of the SF region corresponds to

$$\frac{L(V_c)}{\xi} = \frac{\ln 1/2}{\ln f(\Lambda_{\rm e}(V_c)/\lambda)}\,. \tag{10}$$

Figure 2 displays the stability phase diagram and the cumulative distribution of the critical velocity. Here also we find excellent agreement with the numerical results.

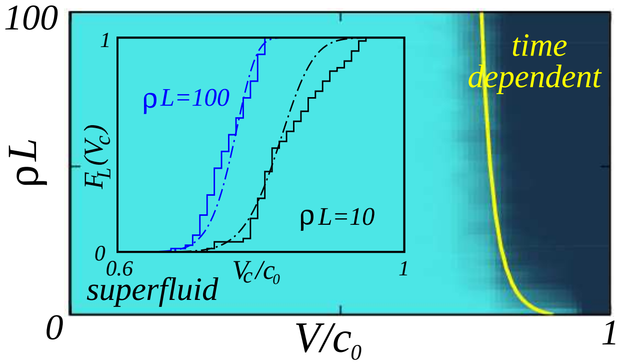

FIG. 2: (Color online) Same as Fig. 1 but for a random potential formed by a sequence of uncorrelated delta peaks ($\lambda = 0.1$, $\rho\xi = 0.1$).

V. DISCUSSSION

There are several experimental possibilities for studying the statistical properties of the boundary of the SF region. The Gaussian disorder with zero average may be implemented experimentally in the case of microfabricated circuits, where the atoms are magnetically guided over a chip [29]. Roughness and disorder in the circuits induce fluctuations along the guide which are typically Lorentzian correlated, with a correlation length ℓ_c which decreases with increasing distance between the guide and the chip [30]. However, the most common type of experimental disorder is the so called speckle potential, generated by a laser beam passing through a diffusing plate [31]. One of the most appropriate set ups seems to be the one used in Ref. [7], where the critical velocity of a trapped Bose-Einstein condensate has been probed by sweeping a laser beam through it. The critical velocity was determined from measurements of the amount of excitations related to the emission of solitons and linear excitations. Similar studies could be done by sweeping the laser beam through a diffusive plate at constant velocity, thus creating a moving speckle potential. Finally, the statistical properties of the SF breaking may also be studied by simply adapting our calculations to the damping of dipole oscillations, along the lines of the recent experiments of Ref. [18].

In conclusion we have stressed the link between the SF critical velocity in presence of disorder and the statistics of extreme events. We have developed simple models in two opposite situations where the disorder is either very smooth or composed of point-like impurities. In both cases the agreement between numerical simulations and our analytical model is very good. We note that, because of the mapping of the statistical properties of the critical velocities to that of extreme events of an uncorrelated sequence, in the limit $L \to \infty$ the distribution of V_c tends to one of the universal distributions of extreme value statistics [32]. A possible extension of this work could be the application to fermionic superfluids in the BCS regime. In that case, the Landau critical velocity is related to the pairing gap, and an equation similar to (5) can be explicitly written down [17].

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